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On the Bernstein Inequality for Rational Functions with a Prescribed Zero

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We extend some results of Giroux and Rahman (*Trans. Amer. Math. Soc.* **193** (1974), 67–98) for Bernstein-type inequalities on the unit circle for polynomials with a prescribed zero at $z=1$ to those for rational functions. These results improve the Bernstein-type inequalities for rational functions. The sharpness of these inequalities is also established. Our approach makes use of the Malmquist–Walsh system of orthogonal rational functions on the unit circle associated with the Lebesgue measure.

Key Words: rational functions; Bernstein inequality; Blaschke product.

1. INTRODUCTION

The classical Bernstein inequalities relate the norm of a polynomial to that of its derivative. Let $\|f\| := \max_{|z|=1} |f(z)|$, the sup-norm on the unit disk. One basic result is as follows: If $P_n(z)$ is a polynomial of degree n such that $\|P_n\| = 1$ then

$$\|P'_n\| \leqslant n, \tag{1.1}$$

and the equality holds only for $P_n(z) = \lambda z^n$, $|\lambda| = 1$. Noting that these extremal polynomials (that is, polynomials such that the equality holds in (1.1)) have all zeros at the origin, Erdős conjectured and later Lax [L] verified that if $\|P_n\| = 1$ and $P_n(z) \neq 0$ in $|z| < 1$ then (1.1) can be replaced by

$$\|P'_n\| \leqslant \frac{n}{2}, \tag{1.2}$$

and the equality holds if all the zeros of $P_n(z)$ lie on $|z| = 1$. R. P. Boas, Jr. asked (see [GR]) what can be said about $\|P'_n\|$ if we assume $P_n(z)$ have

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precisely k zeros in $|z| \geqslant 1$ instead of all the zeros as Erdős did. The sharp inequality for such polynomials has at present not been found. The first step toward a solution was taken by Giroux and Rahman [GR]. Among other things, they proved in that paper that if $\|P_n\| = 1$ and $P_n(1) = 0$, then

$$\|P'_n\| < n - \frac{C}{n}, \tag{1.3}$$

and the inequality is sharp in the sense that there exists a polynomial $p_n(z)$ such that $\|p_n\| = 1$, $p_n(1) = 0$, and

$$\|p'_n\| > n - \frac{c}{n}, \tag{1.4}$$

where $c > 0$ and $C > 0$ are constants not depending on n. The latest development of further results along this line can be found in the papers [FRS, OW].

For rational functions with prescribed poles, sharp Bernstein-type inequalities have been established recently in [BE, LMR]. (We remark that Bernstein-type inequalities for rational functions have appeared in the study of rational approximation problems; for references see [PP]. These inequalities contain some constants which are not optimal.) In particular, the rational function versions analogous to inequalities (1.1) and (1.2) are proved. This naturally leads us to the question of Boas for rational functions instead of polynomials; more precisely, we want to find a rational function version of (1.3). The main result of this paper is an extension of (1.3) to rational functions.

This paper is organized as follows. We introduce the notations in Section 2, state our main results in Section 3. In Section 4 we collect and establish some useful auxiliary results, and in Sections 5 and 6 we prove our main results.

2. NOTATIONS

Let $D_- = \{z \in \mathbf{C} \mid |z| < 1\}$, $D_+ = \{z \in \mathbf{C} \mid |z| > 1\}$ and $T := \{z \in \mathbf{C} \mid |z| = 1\}$. For $\alpha_j \in \mathbf{C}$, $j = 1, 2, ..., n$, let $w(z) = \prod_{j=1}^{n} (1 - \overline{\alpha}_j z)$,

$$B_0(z) = 1 \qquad \text{and} \qquad B_k(z) = \prod_{j=1}^{k} \frac{z - \alpha_j}{1 - \overline{\alpha}_j z}, \qquad k = 1, 2, ..., n.$$

Define the space of rational functions with poles among $1/\overline{\alpha_1}$, $1/\overline{\alpha_2}$, ..., $1/\overline{\alpha_n}$, as

$$\mathcal{R}_n := \mathcal{R}_n(\alpha_1, \alpha_2, ..., \alpha_n) = \left\{ \frac{p(z)}{w(z)} \,\middle|\, p \in \mathcal{P}_n \right\},$$

where $\mathcal{P}_n$ denotes the set of polynomials of degree at most n. Let $\alpha_0 := 0$ and define the Malmquist–Walsh system (cf. [D, W])

$$\varphi_k(z) := \frac{(1-|\alpha_k|^2)^{1/2}\, z}{z-\alpha_k}\, B_k(z), \qquad k = 0, 1, 2, ..., n.$$

Note that $\varphi_0(z) = 1$ and $\varphi_k \in \mathcal{R}_n$ for $k = 0, 1, 2, ..., n$. It is known that (cf. [D, W]) if $\alpha_j \in D_-$, $j = 1, 2, ..., n$, then $\{\varphi_k\}_{k=0}^n$ forms an orthonormal basis for $\mathcal{R}_n$, that is, $\{\varphi_k\}_{k=0}^n \subseteq \mathcal{R}_n$ and

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_k(z)\,\overline{\varphi_j(z)}\,d\theta = \delta_{kj}, \qquad z = e^{i\theta},\ k, j = 0, 1, ..., n. \tag{2.1}$$

From now on, in this paper, we will always assume $\alpha_j \in D_-$ for $j = 1, 2, ..., n$. So, the poles of rational functions in $\mathcal{R}_n$ are all outside of the unit circle $\mathbf{T}$.

Let $S_n(z, \zeta) = \sum_{k=0}^n \varphi_k(z)\,\overline{\varphi_k(\zeta)}$. Then $S_n(z, \zeta)$ is the reproducing kernel of $\mathcal{R}_n$, that is, with $\zeta = e^{i\theta}$,

$$r(z) = \frac{1}{2\pi}\int_0^{2\pi} r(\zeta)\, S_n(z, \zeta)\, d\theta \qquad \text{for every} \quad r \in \mathcal{R}_n. \tag{2.2}$$

For $r \in \mathcal{R}_n$, the following Bernstein inequality has been proved (cf. [BE, LMR]): If $\|r\| = 1$ then

$$|r'(z)| \leqslant |B_n'(z)| \qquad \text{for} \quad z \in \mathbf{T}. \tag{2.3}$$

Furthermore, it is shown that the equality holds only if $r(z) = \lambda B_n(z)$ for some $\lambda \in \mathbf{T}$.

Note that $B_n(z)$ has all its zeros in D_-. Thus, if we put some restrictions on the location of the zeros of $r \in \mathcal{R}_n$, then it may be possible to improve the inequality (2.3). We will show that this is indeed the case: the inequality (2.3) can be strengthened as in the case for polynomials (cf. (1.3)).

3. MAIN RESULTS

Let $a \in [0, 1]$. Following Giroux and Rahman [GR], we define a subset of $\mathcal{R}_n$ by

$$\mathcal{R}_{n,a} := \{r \in \mathcal{R}_n \mid \min_{z \in \mathbf{T}} |r(z)| \leqslant a\}.$$

Thus, $\mathcal{R}_{n,0}$ denotes the subset of all rational functions in $\mathcal{R}_n$ that have at least one zero on $\mathbf{T}$.

Denote $m = m_n := \min_{z \in \mathbf{T}} |B_n'(z)|$ and $M = M_n := \|B_n'\|$. Then, from the properties of Blaschke product $B_n(z)$, it can be verified that (cf. Eq. (4.2) in Section 4)

$$\sum_{j=1}^{n} \frac{1-|\alpha_j|}{1+|\alpha_j|} \leqslant m \leqslant M \leqslant \sum_{j=1}^{n} \frac{1+|\alpha_j|}{1-|\alpha_j|}. \tag{3.1}$$

Later (see (4.4)), we will see that

$$m \leqslant n \leqslant M.$$

THEOREM 3.1. *Assume $r \in \mathcal{R}_{n,a}$ and $\|r\| = 1$. Then*

$$|r'(z)| \leqslant |B_n'(z)| - \frac{(1-a)}{4\pi M} \left\{\frac{m}{M}(1-a) - \sin \frac{m}{M}(1-a)\right\}, \qquad z \in \mathbf{T}. \tag{3.2}$$

Note that, by taking $a = 1$, we obtain the Bernstein inequality (2.3) from (3.2) in Theorem 3.1. As an immediate consequence of Theorem 3.1, using the inequality $x - \sin x \geqslant x^3/8$ for $0 \leqslant x \leqslant 1$ and taking $a = 0$, we infer the following corollary.

COROLLARY 3.2. *Assume $r \in \mathcal{R}_{n,0}$ and $\|r_n\| = 1$. Then*

$$|r'(z)| \leqslant |B_n'(z)| - \frac{m^3}{32\pi M^4}, \qquad z \in \mathbf{T}.$$

In the case when $\alpha_j = 0$ $(j = 1, 2, ..., n)$, $\mathcal{R}_n$ becomes $\mathcal{P}_n$ and $B_n(z) = z^n$. So, $m = M = n$ and Theorem 3.1 reduces to the polynomial inequality (1.3) established by Giroux and Rahman [GR, Theorem 1].

By constructing a near optimal solution, we can verify the sharpness of Theorem 3.1 in the case when $\liminf_{n \to \infty} m/M > 0$. This is given by the following result.

THEOREM 3.3. *Assume* $m \geqslant 2$ *and* $n \geqslant 3$. *Then there exists an absolute constant* $C > 0$ *such that*

$$\max_{\substack{r \in \mathcal{R}_{n,a} \\ \|r\| = 1}} \max_{z \in \mathbf{T}} \left(|r'(z)| - |B_n'(z)| \right) \geqslant -\frac{C}{m} (1-a).$$

Since $\alpha_j \in D_-$, the first inequality in (3.1) implies

$$m > \tfrac{1}{2} \sum_{j=1}^{n} (1 - |\alpha_j|).$$

Hence, Theorem 3.3 has the following consequence. (Compare this with inequality (1.4) for polynomials.)

COROLLARY 3.4. *If* $\sum_{j=1}^{n} (1 - |\alpha_j|) \geqslant 4$ *and* $n \geqslant 3$, *then there exists an absolute constant* $C > 0$ *such that*

$$\max_{\substack{r \in \mathcal{R}_{n,a} \\ \|r\| = 1}} \max_{z \in \mathbf{T}} \left(|r'(z)| - |B_n'(z)| \right) \geqslant -\frac{C}{m} (1-a).$$

We remark that when we consider approximation by rational functions with prescribed poles at $\{\alpha_k\}_{k=1}^{\infty}$ we are more interested in the case when

$$\sum_{k=1}^{\infty} (1 - |\alpha_k|) = \infty, \tag{3.3}$$

since (3.3) is the necessary and sufficient condition for $\bigcup_{n=1}^{\infty} \mathcal{R}_n$ to be dense in the Hardy space on the unit disk, H_p, for $p > 1$, see, e.g., [A]. So, when (3.3) holds, the assumptions in Corollary 3.4 are satisfied if n is large enough.

4. LEMMAS

The main ideas of our proofs are taken from those of Giroux and Rahman's paper [GR] and Rahman and Stenger's paper [RS] (see also the book of Turán [T, Chapter 5 and Appendix A]). In order to carry out those ideas to the new situation, we have to develop some auxiliary results for rational functions.

First, note that the following identity follows from the general Darboux–Christoffel formula for orthogonal rational functions [D], although it can be verified directly.

LEMMA 4.1. *Let $\{\varphi_k\}_{k=0}^n$ and B_n be defined as in Section 2. Then*

$$\sum_{k=0}^{n} \varphi_k(z)\,\overline{\varphi_k(\zeta)} = \frac{1 - B_n(z)\,\overline{B_n(\zeta)}}{1 - z\bar{\zeta}} + B_n(z)\,\overline{B_n(\zeta)} \qquad \textit{for all} \quad z, \zeta \in \mathbf{C}.$$

As a consequence of Lemma 4.1 and the fact that $\varphi_0(z) = 1$, it follows easily that

$$\sum_{k=1}^{n} \varphi_k(z)\,\overline{\varphi_k(\zeta)} = z\bar{\zeta}\,\frac{1 - B_n(z)\,\overline{B_n(\zeta)}}{1 - z\bar{\zeta}}. \tag{4.1}$$

LEMMA 4.2. *For $z \in \mathbf{T}$, there holds*

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1 - B_n(z)\,\overline{B_n(\zeta)}}{1 - z\bar{\zeta}}\right|^2 d\theta = |B_n'(z)|, \qquad (\zeta = e^{i\theta}).$$

Proof. By (4.1), using (2.1), we have (with $\zeta = e^{i\theta}$)

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1 - B_n(z)\,\overline{B_n(\zeta)}}{1 - z\bar{\zeta}}\right|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{k=1}^{n} \varphi_k(z)\,\overline{\varphi_k(\zeta)}\right|^2 d\theta$$
$$= \sum_{k=1}^{n} |\varphi_k(z)|^2 = \sum_{k=1}^{n} \frac{1 - |\alpha_k|^2}{|z - \alpha_k|^2}.$$

It is straightforward to verify that, for $z \in \mathbf{T}$,

$$\sum_{k=1}^{n} \frac{1 - |\alpha_k|^2}{|z - \alpha_k|^2} = \frac{zB_n'(z)}{B_n(z)} = |B_n'(z)|. \tag{4.2}$$

Now, the lemma follows. ∎

Note that, when $|\alpha| < 1$,

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1 - |\alpha|^2}{|z - \alpha|^2}\, d\theta = 1 \qquad (z = e^{i\theta}).$$

So, we have, by using (4.2),

$$\frac{1}{2\pi}\int_0^{2\pi} |B'(z)|\, d\theta = n \qquad (z = e^{i\theta}). \tag{4.3}$$

From this, it follows that

$$m = \min_{z \in \mathbf{T}} |B'(z)| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |B'(z)|\, d\theta = n \leqslant \max_{z \in \mathbf{T}} |B'(z)| = M. \tag{4.4}$$

The following integral representation formula is simple and very useful. It is a generalization of an integral representation for polynomials used in the proof of Theorem 1 in Giroux and Rahman [GR, p 88, line 5].

LEMMA 4.3. *For* $z \in \mathbf{T}$ *and* $r \in \mathcal{R}_n$,

$$r'(z) = \frac{1}{2\pi} \int_0^{2\pi} r(\zeta) \bar{\zeta} \left\{ \frac{1 - B_n(z)\, \overline{B_n(\zeta)}}{1 - z\bar{\zeta}} \right\}^2 d\theta, \qquad \zeta = e^{i\theta}. \tag{4.5}$$

Proof. On differentiating both sides of the Eq. (2.2) with respect to z we obtain

$$r'(z) = \frac{1}{2\pi} \int_0^{2\pi} r(\zeta) \frac{\partial}{\partial z} S_n(z, \zeta)\, d\theta.$$

Now, using Lemma 4.1 and relation (4.2) and writing

$$\beta := \sum_{k=1}^{n} \frac{1 - |\alpha_k|^2}{|z - \alpha_k|^2},$$

we have, for $z \in \mathbf{T}$ and $\zeta \in \mathbf{T}$,

$$\begin{aligned} \frac{\partial}{\partial z} S_n(z, \zeta) &= \frac{-(1 - z\bar{\zeta})\, B_n'(z)\, \overline{B_n(\zeta)} + (1 - B_n(z)\, \overline{B_n(\zeta)})\bar{\zeta}}{(1 - z\bar{\zeta})^2} + B_n'(z)\, \overline{B_n(\zeta)} \\ &= \frac{-(1 - z\bar{\zeta})\, \bar{z} B_n(z)\, \beta \overline{B_n(\zeta)} + (1 - B_n(z)\, \overline{B_n(\zeta)})\bar{\zeta}}{(1 - z\bar{\zeta})^2} + \frac{B_n'(z)}{B_n(\zeta)}. \end{aligned}$$

So, for $z \in \mathbf{T}$ and $\zeta \in \mathbf{T}$, the first term of the above expression equals

$$\begin{aligned} &\bar{\zeta} \frac{(1 - \bar{z}\zeta)\, B_n(z)\, \beta \overline{B_n(\zeta)} + 1 - B_n(z)\, \overline{B_n(\zeta)}}{(1 - z\bar{\zeta})^2} \\ &\quad = \bar{\zeta} \frac{(1 - B_n(z)\, \overline{B_n(\zeta)})^2 + B_n(z)\, \overline{B_n(\zeta)}[1 + (1 - \bar{z}\zeta)\beta - B_n(z)\, \overline{B_n(\zeta)}]}{(1 - z\bar{\zeta})^2} \\ &\quad = \bar{\zeta} \left\{ \frac{1 - B_n(z)\, \overline{B_n(\zeta)}}{1 - z\bar{\zeta}} \right\}^2 + \frac{\zeta B_n(z)}{B_n(\zeta)} \frac{P(\zeta)}{(\zeta - z)^2}, \end{aligned}$$

where $P(\xi) := 1 + (1 - \bar{z}\xi)\, \beta - B_n(z)/B_n(\xi)$. For $z \in \mathbf{T}$, it is easy to see that $P(z) = P'(z) = 0$. Also, note that the poles of $P(\xi)$ are $\{\alpha_k\}_{k=1}^n$, so

$$P(\xi) = \frac{(\xi - z)^2\, q_{n-1}(\xi)}{\prod_{k=1}^n (\xi - \alpha_k)} \qquad \text{for} \quad q_{n-1} \in \mathcal{P}_{n-1}.$$

Thus, for $z \in \mathbf{T}$ and $\zeta \in \mathbf{T}$,

$$\frac{\partial}{\partial z} S_n(z, \zeta) = \bar{\zeta} \left\{ \frac{1 - B_n(z)\, \overline{B_n(\zeta)}}{1 - z\zeta} \right\}^2 + \frac{\zeta B_n(z)\, q_{n-1}(\zeta)}{B_n(\zeta) \prod_{k=1}^{n} (\zeta - \alpha_k)} + \frac{B_n'(z)}{B_n(\zeta)}$$

$$=: K_1(\zeta) + K_2(\zeta) + K_3(\zeta).$$

Note that, for $0 \leqslant k \leqslant n$, $\varphi_k(\zeta)\, K_2(\zeta)$ is a rational function whose numerator and denominator are each of degree $2n - k + 1$, and whose poles are all in D_- (in fact, these poles are located at $\{\alpha_k\}_{k=0}^{n}$ with proper multiplicities). So, on applying Cauchy's theorem (to $D_+ \cup \{\infty\}$) we have

$$\int_0^{2\pi} \varphi_k(\zeta)\, K_2(\zeta)\, d\theta = 0.$$

Similarly, we have

$$\int_0^{2\pi} \varphi_k(\zeta)\, K_3(\zeta)\, d\theta = 0.$$

Therefore, for $z \in \mathbf{T}$,

$$\int_0^{2\pi} \varphi_k(\zeta) \frac{\partial}{\partial z} S_n(z, \zeta)\, d\theta = \int_0^{2\pi} \varphi_k(\zeta)\, K_1(\zeta)\, d\theta \qquad (\zeta = e^{i\theta}),$$

for $k = 0, 1, ..., n$. Since $\{\varphi_k\}_{k=0}^{n}$ is a base of $\mathscr{R}_n$, so, for $z \in \mathbf{T}$ and $r \in \mathscr{R}_n$,

$$\int_0^{2\pi} r(\zeta) \frac{\partial}{\partial z} S_n(z, \zeta)\, d\theta = \int_0^{2\pi} r(\zeta)\, K_1(\zeta)\, d\theta,$$

which implies (4.5). ∎

The next lemma gives us a local estimate of rational functions in $\mathscr{R}_n$. We recall that $M := \|B_n'\|$.

Lemma 4.4. *If* $r \in \mathscr{R}_n$, $|r(e^{i\psi_0})| = a$ *for some* $\psi_0 \in [0, 2\pi]$ *and* $a \in [0, 1]$, *and* $\|r\| = 1$, *then*

$$|r(e^{i\theta})| \leqslant \frac{1 + a}{2} \qquad \textit{for} \quad |\theta - \psi_0| \leqslant \frac{1 - a}{M}.$$

Proof. Without loss of generality, we assume $\psi_0 = 0$. We follow the argument in [GR, Lemma 3]. Assume first that r has no zero in D_- and write

$$r(e^{i\theta}) - r(1) = \int_1^{e^{i\theta}} r'(z)\, dz,$$

where the integration is on the line segment from 1 to $e^{i\theta}$. So

$$|r(e^{i\theta})| \leqslant a + \|r'\| \; |e^{i\theta} - 1| = a + 2\,\|r'\| \left|\sin \frac{\theta}{2}\right|.$$

Now, by [LMR, Theorem 3], since r has no zero in D_-,

$$\|r'\| \leqslant \frac{\|B_n'(z)\|}{2}\,\|r\| = \frac{M}{2}.$$

Thus,

$$|r(e^{i\theta})| \leqslant a + \left|\sin \frac{\theta}{2}\right| M \leqslant a + \frac{|\theta|}{2}\,M.$$

Hence, if $|\theta| \leqslant (1-a)/M$,

$$|r(e^{i\theta})| \leqslant a + \frac{1-a}{2M}\,M = \frac{1+a}{2},$$

which proves the lemma when r has no zero in D_-.

The case when r has zero in D_- can be proved by considering $r(z)\,b(z)$ where $b(z)$ is the Blaschke product whose poles are at those zeros of r that are in D_-. ▮

Lemma 4.5. *For $z \in \mathbf{T}$ and $\theta_1 \leqslant \theta_2$ with $|\theta_1 - \theta_2| \leqslant 2\pi/M$, we have*

$$\int_{\theta_1}^{\theta_2} \left|\frac{1 - B_n(z)\,\overline{B_n(e^{i\theta})}}{1 - ze^{-i\theta}}\right|^2 d\theta \geqslant \frac{1}{M}\left\{\frac{m(\theta_2-\theta_1)}{2} - \sin\frac{m(\theta_2-\theta_1)}{2}\right\}.$$

Proof. Let $\theta_0 \in [0, 2\pi)$ be chosen so that $B_n(e^{i\theta_0}) = 1$. Define

$$\gamma(\theta) := \int_{\theta_0}^{\theta} |B_n'(e^{i\theta})|\,d\theta \qquad \text{for all real } \theta. \tag{4.6}$$

Then, $d\gamma(\theta)/d\theta = |B_n'(e^{i\theta})| \in [m, M]$ for al real θ. This and the mean value theorem imply

$$\gamma(\theta_2) - \gamma(\theta_1) \leqslant M(\theta_2 - \theta_1) \tag{4.7}$$

and

$$\gamma(\theta_2) - \gamma(\theta_1) \geqslant m(\theta_2 - \theta_1). \tag{4.8}$$

Since $B_n(e^{i\theta_o}) = e^{i\gamma(\theta_o)} = 1$ and, by (4.2) and a direct computation,

$$\frac{d}{d\theta} \frac{e^{i\gamma(\theta)}}{B_n(e^{i\theta})} = 0 \qquad \text{for all } \theta,$$

we have

$$B_n(e^{i\theta}) = e^{i\gamma(\theta)} \tag{4.9}$$

for all real θ. Using this representation of $B_n(e^{i\theta})$ in terms of $\gamma(\theta)$, with $J := [\theta_1, \theta_2]$ and $z = e^{i\psi}$, we can write

$$\begin{aligned}
&\int_J \left| \frac{1 - B_n(e^{i\psi}) \overline{B_n(e^{i\theta})}}{1 - e^{i\psi} e^{-i\theta}} \right|^2 d\theta \\
&\quad \geqslant \frac{1}{4} \int_J |1 - B_n(e^{i\psi}) \overline{B_n(e^{i\theta})}|^2 \, d\theta \\
&\quad = \frac{1}{4} \int_J |1 - e^{-i(\gamma(\theta) - \gamma(\psi))}|^2 \, d\theta = \frac{1}{4} \int_J 2[1 - \cos(\gamma(\theta) - \gamma(\psi))] \, d\theta \\
&\quad \geqslant \frac{1}{2M} \int_{\theta \in J} [1 - \cos(\gamma(\theta) - \gamma(\psi))] \, d\gamma(\theta) \\
&\quad = \frac{1}{2M} \left\{ \gamma(\theta_2) - \gamma(\theta_1) - 2 \cos\left(\frac{\gamma(\theta_1) - \gamma(\theta_2)}{2} - \gamma(\psi) \right) \sin \frac{\gamma(\theta_2) - \gamma(\theta_1)}{2} \right\}.
\end{aligned}$$

From (4.7) and the assumption that $\theta_2 - \theta_1 \leqslant 2\pi/M$, we find $\sin[(\gamma(\theta_2) - \gamma(\theta_1))/2] \geqslant 0$, and so the expression in the curly brackets is greater than

$$\gamma(\theta_2) - \gamma(\theta_1) - 2 \sin \frac{\gamma(\theta_2) - \gamma(\theta_1)}{2}.$$

Using (4.8) and the fact that $x - 2 \sin x/2$ is an increasing function of x, we obtain

$$\gamma(\theta_2) - \gamma(\theta_1) - 2 \sin \frac{\gamma(\theta_2) - \gamma(\theta_1)}{2} \geqslant m(\theta_2 - \theta_1) - 2 \sin \frac{m(\theta_2 - \theta_1)}{2}.$$

Therefore

$$\int_{\theta_1}^{\theta_2} \left| \frac{1 - B_n(z) \overline{B_n(e^{i\theta})}}{1 - z e^{-i\theta}} \right|^2 d\theta \geqslant \frac{1}{M} \left\{ \frac{m(\theta_2 - \theta_1)}{2} - \sin \frac{m(\theta_2 - \theta_1)}{2} \right\}.$$

This completes the proof. ∎

We will need the following properties of the Blaschke product in the proof of Theorem 3.3 in Section 6. Recall that $m = \min_{z \in \mathbf{T}} |B_n'(z)|$.

LEMMA 4.6. *Assume that* $m \geqslant 2$ *and* $\zeta \in \mathbf{T}$. *Then*

(i) *there exist* $\theta_1, \theta_2, ..., \theta_n$,

$$-\pi \leqslant \theta_1 < \theta_2 < \cdots < \theta_n < \pi,$$

such that $B_n(e^{i\theta_k}) = B_n(\zeta)$ *with* $\zeta = e^{i\theta_j}$ *for some* $j \in \{1, 2, ..., n\}$;

(ii) *with* $\theta_{n+1} := \theta_1 + 2\pi$, *we have* $|\theta_{l+1} - \theta_l| \leqslant \pi$, $l = 1, 2, ..., n$;

(iii) *for* γ *defined as in* (4.6) *and for* j *in* (*i*),

$$\gamma(\theta_k) - \gamma(\theta_j) = 2(k-j)\pi, \qquad k = 1, 2, ..., n. \tag{4.10}$$

Proof. Note that, by the definition of $\gamma(\theta)$ and (4.4), $\gamma(\theta)$ is strictly increasing on $[-\pi, \pi]$ and $\gamma(\pi) - \gamma(-\pi) = 2n\pi$. Using the intermediate value theorem, we can find n values of θ such that

$$-\pi \leqslant \theta_1 < \theta_2 < \cdots < \theta_n < \pi$$

and, with $\zeta = e^{i\theta_j}$ for some $j \in \{1, 2, ..., n\}$,

$$\gamma(\theta_k) = \gamma(\theta_j) + 2(k-j)\pi, \qquad k = 1, 2, ..., n.$$

Note that

$$\begin{aligned} 2\pi = \gamma(\theta_{l+1}) - \gamma(\theta_l) &= \gamma'(\theta_l')(\theta_{l+1} - \theta_l) \\ &\geqslant m(\theta_{l+1} - \theta_l) \geqslant 2(\theta_{l+1} - \theta_l). \end{aligned}$$

For $\theta_1, \theta_2, ..., \theta_n$ chosen above, the conclusions of the lemmas are satisfied. ∎

The following elementary statement is also needed in our proof of Theorem 3.3 (for comparison, see relation (A.1.1.) and (A.1.2) in [T]).

LEMMA 4.7. *Let*

$$f(x) = \begin{cases} \dfrac{1}{(\sin x)^2} - \dfrac{1}{x^2}, & x \neq 0 \\[2mm] \dfrac{1}{3}, & x = 0. \end{cases}$$

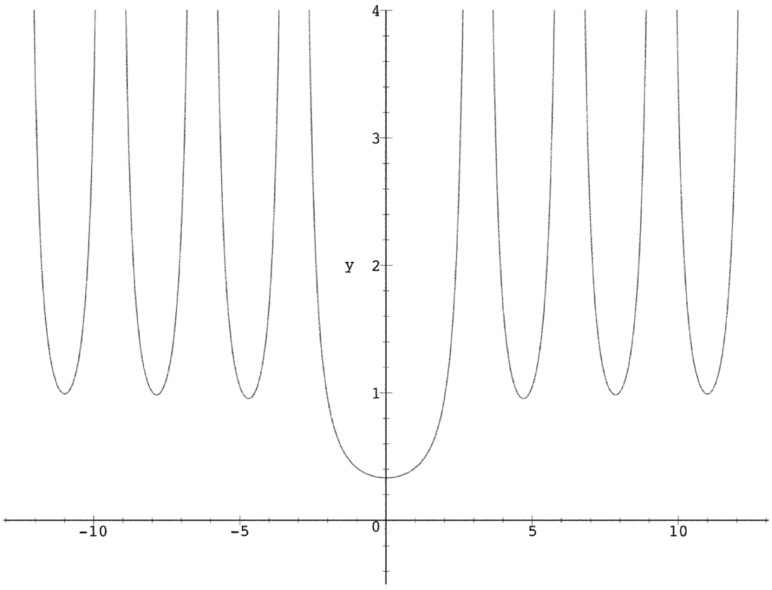

FIG. 4.1. $f(x)$ on $[-4\pi, 4\pi]$.

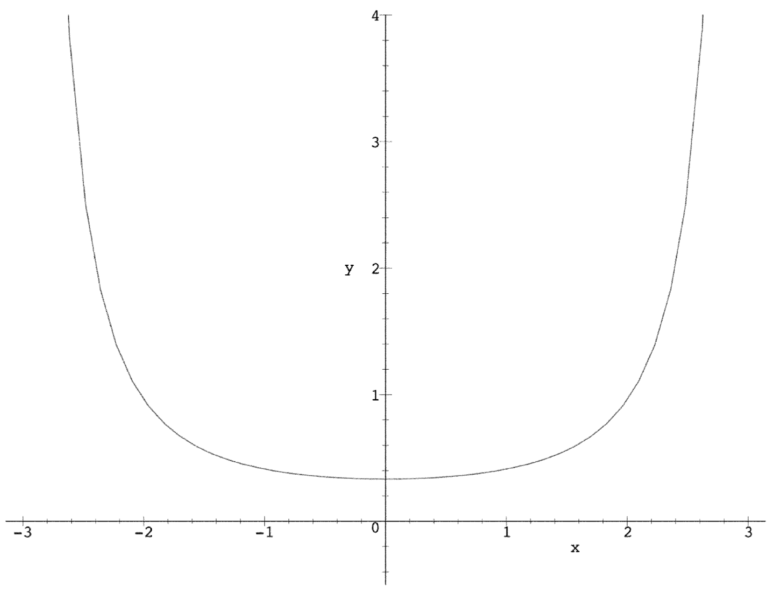

FIG. 4.2. $f(x)$ on $[-\pi, \pi]$.

Then f is continuous (for all $x \neq k\pi$, $k = \pm 1, \pm 2, \ldots$), even, and increasing for $(0, \pi]$.

Furthermore

$$\min_{\substack{x \neq k\pi \\ k = \pm 1, \pm 2, \ldots}} f(x) = \tfrac{1}{3}.$$

The graph of the function is shown in Figs. 4.1 and 4.2. A proof of this lemma can be obtained by using series representation and properties of alternating series.

5. PROOF OF THEOREM 3.1

Assume $r(e^{i\psi_0}) = \min_{z \in \mathbf{T}} |r(z)| = a$ and $\|r\| = 1$. By Lemmas 4.3 and 4.4, with $J := \{\theta \colon |\theta - \psi_0| \leqslant (1-a)/M\}$ and $J' := [\psi_0 - \pi, \psi_0 + \pi] \backslash J$, we have

$$\begin{aligned}
|r'(z)| &\leqslant \frac{1}{2\pi} \int_0^{2\pi} |r(e^{i\theta})| \left| \frac{1 - B_n(z)\,\overline{B_n(e^{i\theta})}}{1 - ze^{-i\theta}} \right|^2 d\theta = \frac{1}{2\pi} \left(\int_{J'} + \int_J \right) \\
&\leqslant \frac{1}{2\pi} \int_{J'} \left| \frac{1 - B_n(z)\,\overline{B_n(e^{i\theta})}}{1 - ze^{-i\theta}} \right|^2 d\theta + \frac{1}{2\pi} \int_J \frac{1+a}{2} \left| \frac{1 - B_n(z)\,\overline{B_n(e^{i\theta})}}{1 - ze^{-i\theta}} \right|^2 d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1 - B_n(z)\,\overline{B_n(e^{i\theta})}}{1 - ze^{-i\theta}} \right|^2 d\theta - \frac{1-a}{4\pi} \int_J \left| \frac{1 - B_n(z)\,\overline{B_n(e^{i\theta})}}{1 - ze^{-i\theta}} \right|^2 d\theta \\
&= |B_n'(z)| - \frac{1-a}{4\pi} \int_J \left| \frac{1 - B_n(z)\,\overline{B_n(e^{i\theta})}}{1 - ze^{-i\theta}} \right|^2 d\theta.
\end{aligned}$$

Here in the last equality, we have used Lemma 4.2. Now, using Lemma 4.5 for $[\theta_1, \theta_2] = J$, we get

$$\int_{|\theta - \psi_0| \leqslant (1-a)/M} \left| \frac{1 - B_n(z)\,\overline{B_n(e^{i\theta})}}{1 - ze^{-i\theta}} \right|^2 d\theta \geqslant \frac{1}{M} \left\{ \frac{m}{M}(1-a) - \sin \frac{m}{M}(1-a) \right\}.$$

Using this in the previous inequality, we obtain

$$|r'(z)| \leqslant |B_n'(z)| - \frac{1-a}{4\pi M} \left\{ \frac{m}{M}(1-a) - \sin \frac{m}{M}(1-a) \right\},$$

which is the desired result. ∎

6. PROOF OF THEOREM 3.3

This proof is based on the construction of a rational function $t_n \in \mathcal{R}_{n,a}$ so that an upper bound of $|t_n'(z)| - |B_n'(z)|$ can be calculated at certain points. We begin by constructing $r_n \in R_{n,0}$ for which we have $|r_n'(z_0)| \leqslant C/m$ at some $z_0 \in \mathbf{T}$ for some absolute constant $C > 0$. Using r_n we construct $t_n \in \mathcal{R}_{n,a}$ and show that $|t_n'(z)| - |B_n'(z)| \geqslant -(C/m)(1-a)$.

Let $\zeta \in \mathbf{T}$ satisfy $|B_n'(\zeta)| = M$, and let

$$C_n(z) := \frac{B_n(z) - B_n(\zeta)}{z - \zeta}.$$

Then $C_n \in \mathcal{R}_n$, $C_n(\zeta) = B_n'(\zeta)$, and

$$|C_n(z)| = \frac{1}{|z-\zeta|} \left| \int_\zeta^z B_n'(\xi)\, d\xi \right| \leqslant \max_{\xi \in \mathbf{T}} |B_n'(\xi)| = M$$

for $z \in \mathbf{T}$, where the path of integration is taken along the chord from ζ to z. Thus,

$$|w(e^{i\theta})|^2 \, (M^2 - |C_n(e^{i\theta})|^2)$$

is a non-negative real trigonometric polynomial (in θ) of degree at most n. By a theorem of Fejér (see, for example [S, Theorem 1.2.2]), there exists a unique algebraic polynomial $\rho(z)$ of degree at most n such that

(i) $|\rho(z)|^2 = |w(z)|^2 \, (M^2 - |C_n(z)|^2)$, $z \in \mathbf{T}$,

(ii) $\rho(0) > 0$,

(iii) $\rho(z) \neq 0$ for all $z \in D_-$.

Define

$$r_n(z) = \frac{\rho(z)}{Mw(z)}.$$

Then $r_n(z) \in \mathcal{R}_{n,0} (|r_n(\zeta)|^2 = 1 - |C_n(\zeta)|^2/M^2 = 0)$. Since $r_n(z) \neq 0$ for $z \in D_-$, $r_n(0) > 0$, and $|r_n(z)| \leqslant 1$ for $z \in \mathbf{T}$, we can use a representation for H_∞ functions to write (cf. [Du]) or directly verify that

$$r_n(z) = \exp\left\{ \frac{1}{4\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |r_n(e^{i\theta})|^2 \, d\theta \right\}$$

for $z \in D_-$. So

$$r_n'(z) = r_n(z) \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}}{(e^{i\theta} - z)^2} \log |r_n(e^{i\theta})|^2 \, d\theta \right\} \tag{6.1}$$

for $z \in D_-$. Next, we show that (6.1) holds for some suitably chosen points on $\mathbf{T}$. Note that

$$|r_n(e^{i\theta})|^2 = 1 - \frac{|C_n(e^{i\theta})|^2}{M^2} = 1 - \frac{|B_n(e^{i\theta}) - B_n(\zeta)|^2}{M^2 \, |e^{i\theta} - \zeta|^2}. \tag{6.2}$$

It can be verified that

$$|r_n(e^{i\theta})| = 1 \qquad \text{if and only if} \quad B_n(e^{i\theta}) = B_n(\zeta) \text{ and } e^{i\theta} \neq \zeta.$$

Applying Lemma 4.6, there exist

$$-\pi \leqslant \theta_1 < \theta_2 < \cdots < \theta_n < \pi$$

such that $B_n(e^{i\theta_k}) = B_n(\zeta)$ with $\zeta = e^{i\theta_j}$ for some j. It then follows that

$$\log |r_n(e^{i\theta})|^2 \qquad (\leqslant 0)$$

has a zero of even multiplicity at each $\theta = \theta_k$ $(k \neq j)$. Let us fix $k \in \{1, 2, ..., n\} \setminus \{j\}$; then

$$\frac{e^{i\theta}}{(e^{i\theta} - e^{i\theta_k})^2} \log |r_n(e^{i\theta})|^2$$

is continuous at $\theta = \theta_k$. Hence, we can find an $\varepsilon > 0$ small enough so that the function

$$\frac{e^{i\theta}}{(e^{i\theta} - ue^{i\theta_k})^2} \log |r_n(e^{i\theta})|^2$$

is bounded for all $(\theta, u) \in [\theta_k - \varepsilon, \theta_k + \varepsilon] \times [0, 1]$. Therefore, we can take $z = ue^{i\theta_k}$ and let $u \to 1^-$ in (6.1) to obtain the formula $z = e^{i\theta_k}$ by using the bounded convergence theorem. This establishes the validity of (6.1) at $z = e^{i\theta_k}$.

Now, we estimate $|r_n'(e^{i\theta_k})|$ by (6.1). Let

$$I := \int_0^{2\pi} \frac{4e^{i\theta}}{(e^{i\theta} - e^{i\theta_k})^2} \log |r_n(e^{i\theta})|^2 \, d\theta.$$

We next show that the integral I is of order $1/m$ as $n \to \infty$, where $m = m_n = \min_{z \in \mathbf{T}} |B_n'(z)|$ as defined in Section 3.

On using (6.2) and writing $B_n(e^{i\theta}) = e^{i\gamma(\theta)}$ as in the proof of Lemma 4.5, we derive[1]

$$|I| = -\int_0^{2\pi} \frac{1}{[\sin(\theta-\theta_k)/2)]^2} \log \left\{ 1 - \left(\frac{\sin(\gamma(\theta)-\gamma(\theta_j))/2}{M \sin(\theta-\theta_j)/2} \right)^2 \right\} d\theta. \quad (6.3)$$

Note that, for simplicity of notation, we write $\sin \phi/2$ instead of $\sin(\phi/2)$. Let $\delta \in (0, \pi)$ be some positive constant to be determined (indeed, we will show $\delta = \pi/6$ is a valid choice), and for θ_j as above (that is, $\zeta = e^{i\theta_j}$) define

$$I_1 := \int_{\theta_j - \pi}^{\theta_j + \pi} \log \left\{ 1 - \left(\frac{\sin(\gamma(\theta)-\gamma(\theta_j))/2}{M \sin(\theta-\theta_j)/2} \right)^2 \right\}^{-1} d\theta$$

and

$$I_2 := \int_{|\theta-\theta_k| \leqslant \delta} \frac{1}{[\sin(\theta-\theta_k)/2]^2} \log \left\{ 1 - \left(\frac{\sin(\gamma(\theta)-\gamma(\theta_j))/2}{M \sin(\theta-\theta_j)/2} \right)^2 \right\}^{-1} d\theta. \quad (6.4)$$

Then, it is easy to see that

$$|I| \leqslant I_1/\sin^2(\delta/2) + I_2. \quad (6.5)$$

To estimate I_1, we use a method in the book of Turán who attributes it to Rahman and Stenger (see [T, p. 52]). For this purpose, we need Lemma 4.7. From this lemma, we have

$$\frac{1}{(\sin x)^2} - \frac{1}{x^2} \leqslant 1 - \frac{4}{\pi^2} \qquad \text{for} \quad |x| \leqslant \frac{\pi}{2}$$

and

$$\frac{1}{(\sin x)^2} - \frac{1}{x^2} \geqslant \frac{1}{3} \qquad \text{for all } x.$$

Thus, since $M \geqslant m \geqslant 2 > 3(1-4/\pi^2)$,

$$\frac{1}{[\sin(\gamma(\theta+\theta_j)-\gamma(\theta_j))/2]^2} - \frac{1}{[(\gamma(\theta+\theta_j)-\gamma(\theta_j))/2]^2}$$
$$\geqslant \frac{1}{3} \geqslant \frac{1}{M}\left(1 - \frac{4}{\pi^2}\right) \geqslant \frac{1}{M}\left\{ \frac{1}{(\sin \theta/2)^2} - \frac{1}{(\theta/2)^2} \right\}$$

[1] The truth of equation (6.3) is the main reason that we use the integral representation (6.1) instead of (4.5).

for $|\theta| \leqslant \pi$. Hence

$$\frac{1}{[\sin(\gamma(\theta+\theta_j)-\gamma(\theta_j))/2]^2}-\frac{1}{[(\gamma(\theta+\theta_j)-\gamma(\theta_j))/2]^2}$$
$$\geqslant \frac{1}{M}\left\{\frac{1}{(\sin\theta/2)^2}-\frac{1}{(\theta/2)^2}\right\}.$$

Now, using the method of Rahman and Stenger as described in [T], together with the fact that $|\gamma(\theta+\theta_j)-\gamma(\theta_j)| \leqslant M|\theta|$, we can infer the inequality

$$1-\left(\frac{\sin(\gamma(\theta+\theta_j)-\gamma(\theta_j))/2}{M\sin\theta/2}\right)^2$$
$$\geqslant \left(1-\frac{1}{M}\right)\left\{1-\left(\frac{\sin(\gamma(\theta+\theta_j)-\gamma(\theta_j))/2}{(\gamma(\theta+\theta_j)-\gamma(\theta_j))/2}\right)^2\right\}.$$

So

$$0 \leqslant I_1 \leqslant -\int_0^{2\pi} \log\left\{\left(1-\frac{1}{M}\right)\left[1-\left(\frac{\sin(\gamma(\theta+\theta_j)-\gamma(\theta_j))/2}{(\gamma(\theta+\theta_j)-\gamma(\theta_j))/2}\right)^2\right]\right\} d\theta$$
$$= -2\pi\log\left(1-\frac{1}{M}\right)-2\int_{[\gamma(-\pi+\theta_j)-\gamma(\theta_j)]/2}^{[\gamma(\pi+\theta_j)-\gamma(\theta_j)]/2} \log\left\{1-\left(\frac{\sin\alpha}{\alpha}\right)^2\right\}\frac{d\alpha}{\gamma'(\theta+\theta_j)}$$
$$\leqslant \frac{4\pi}{M}-\frac{2}{m}\int_{-\infty}^{\infty}\log\left\{1-\left(\frac{\sin\alpha}{\alpha}\right)^2\right\}d\alpha = O\left(\frac{1}{m}\right).$$

Here, in the last inequality, we have used the fact that

$$-\log(1-x) \leqslant 2x \qquad \text{if} \quad x \in [0, 1/2] \tag{6.6}$$

with $x = 1/M$ for $M \geqslant 2$. (In fact, $M \geqslant n \geqslant 3$ by the assumption of the theorem.)

Next, we estimate I_2 defined in (6.4). By using (ii) in Lemma 4.6, we first choose θ_k such that

$$\min(|\theta_k-\theta_j|,\ 2\pi-|\theta_k-\theta_j|) \geqslant \frac{\pi}{2}. \tag{6.7}$$

Then, with $|\alpha| \leqslant \pi/6$, we claim that

$$\frac{\pi}{6} \leqslant \frac{|\alpha+\theta_k-\theta_j|}{2} \leqslant \pi-\frac{\pi}{6}. \tag{6.8}$$

In fact, note that

$$\big|\, |\alpha+\theta_k-\theta_j| - |\theta_k-\theta_j| \,\big| \leqslant |\alpha| \leqslant \frac{\pi}{6},$$

so we have

$$|\theta_k-\theta_j| - \frac{\pi}{6} \leqslant |\alpha+\theta_k-\theta_j| \leqslant |\theta_k-\theta_j| + \frac{\pi}{6}.$$

Hence, by using (6.7),

$$|\alpha+\theta_k-\theta_j| \leqslant |\theta_k-\theta_j| + \frac{\pi}{6} \leqslant 2\pi - \frac{\pi}{2} + \frac{\pi}{6} = 2\pi - \frac{\pi}{3} \tag{6.9}$$

and

$$|\alpha+\theta_k-\theta_j| \geqslant |\theta_k-\theta_j| - \frac{\pi}{6} \geqslant \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}. \tag{6.10}$$

Now, our claim (6.8) follows from (6.9) and (6.10).

Thus, when $|\alpha| \leqslant \pi/6$, we have

$$\left| \sin \frac{\alpha+\theta_k-\theta_j}{2} \right| \geqslant \sin \frac{\pi}{6} = \frac{1}{2}.$$

From now on, we will let $\delta = \pi/6$. With $\alpha = \theta - \theta_k$ we can write I_2 as

$$I_2 = \int_{|\alpha| \leqslant \pi/6} \frac{1}{\sin^2 \alpha/2} \log \left\{ 1 - \left(\frac{\sin(\gamma(\alpha+\theta_k) - \gamma(\theta_k))/2}{M \sin(\alpha+\theta_k-\theta_j)/2} \right)^2 \right\}^{-1} d\alpha.$$

Then

$$\begin{aligned} I_2 &\leqslant \int_{|\alpha| \leqslant \pi/6} \frac{1}{\sin^2 \alpha/2} \log \left\{ 1 - \left(\frac{\sin(\gamma(\alpha+\theta_k) - \gamma(\theta_k))/2}{M \frac{1}{2}} \right)^2 \right\}^{-1} d\alpha \\ &=: I_{21} + I_{22}, \end{aligned}$$

where I_{21} is the integral over $\pi/(6M) \leqslant |\alpha| \leqslant \pi/6$, and I_{22} is the integral over $|\alpha| \leqslant \pi/(6M)$.

Then, by $|\sin x| \leqslant 1$ and $\sin x \geqslant (2/\pi)\, x$ $(0 \leqslant x \leqslant \pi/2)$,

$$I_{21} \leqslant \int_{\pi/6M \leqslant |\alpha| \leqslant \pi/6} \frac{\pi^2}{\alpha^2} \log\left(1 - \frac{4}{M^2}\right)^{-1} d\alpha$$
$$= 2\pi^2 \log\left(1 - \frac{4}{M^2}\right)^{-1} \left(\frac{6M}{\pi} - \frac{6}{\pi}\right).$$

Since $M \geqslant n \geqslant 3$, so $4/M^2 \leqslant 1/2$, and (6.6) implies

$$\log\left(1 - \frac{4}{M^2}\right)^{-1} \leqslant \frac{8}{M^2}.$$

It then follows that

$$I_{21} \leqslant 2\pi^2 \frac{8}{M^2} \left(\frac{6M}{\pi} - \frac{6}{\pi}\right) = O\left(\frac{1}{M}\right).$$

To estimate I_{22}, we proceed as follows:

$$I_{22} = \int_{|\alpha| \leqslant \pi/6M} \frac{1}{\sin^2 \alpha/2} \log\left\{1 - \left(\frac{2\sin(\gamma(\alpha+\theta_k) - \gamma(\theta_k))/2}{M}\right)^2\right\}^{-1} d\alpha$$
$$\leqslant \int_{|\alpha| \leqslant \pi/6M} \frac{\pi^2}{\alpha^2} \log\left\{1 - \left(\frac{\gamma(\alpha+\theta_k) - \gamma(\theta_k)}{M}\right)^2\right\}^{-1} d\alpha.$$

Using the mean value theorem, for some θ^* between θ_k and $\alpha + \theta_k$, we can write the last integral as

$$\int_{|\alpha| \leqslant \pi/6M} \frac{\pi^2}{\alpha^2} \log\left\{1 - \left(\frac{\gamma'(\theta^*)\alpha}{M}\right)^2\right\}^{-1} d\alpha \leqslant \int_{|\alpha| \leqslant \pi/6M} \frac{\pi^2}{\alpha^2} \log(1-\alpha^2)^{-1}\, d\alpha,$$

by the fact that $|\gamma'(\theta^*)| \leqslant M$. Therefore,

$$I_{22} \leqslant \int_{|\alpha| \leqslant \pi/6M} \frac{\pi^2}{\alpha^2} \log(1-\alpha^2)^{-1}\, d\alpha$$
$$\leqslant 2\,\frac{\pi}{6M}\,\frac{\pi^2}{\alpha^2} \log(1-\alpha^2)^{-1}\Big|_{\alpha = \pi/6M}$$
$$= \frac{\pi^3}{3M}\left(\frac{6M}{\pi}\right)^2 \log\left\{1 - \left(\frac{\pi}{6M}\right)^2\right\}^{-1} = O\left(\frac{1}{M}\right).$$

Thus, $I_2 \leqslant I_{21} + I_{22} = O(1/M)$.

Now, we have established that, if θ_k is chosen such that

$$\min(|\theta_k - \theta_j|, 2\pi - |\theta_k - \theta_j|) \geqslant \frac{\pi}{2}$$

(Such k's always exist according to Lemma 4.6 (ii)), then there exists an absolute constant $C > 0$ such that (cf. (6.5))

$$|r_n'(e^{i\theta_k})| \leqslant \frac{C}{m}.$$

Intuitively, we choose k such that $e^{i\theta_k}$ is furthest away from $e^{i\theta_j}$ among $e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_n}$.

Let $s_n(z) = B_n(z)\,\overline{r_n(1/\bar{z})}$. Then for $z \in \mathbf{T}$ we have

$$s_n'(z) = B_n'(z)\overline{r_n(z)} - B_n(z)\,\overline{z^2 r_n'(z)}.$$

Hence

$$|s_n'(e^{i\theta_k})| \geqslant |B_n'(e^{i\theta_k})| - \frac{C}{m} \tag{6.11}$$

and $s_n \in \mathscr{R}_{n,0}$. For $a \in [0, 1]$, define $t_n(z) = \mu a B_n(z) + (1-a)\, s_n(z)$ for some $\mu \in \mathbf{T}$ to be chosen later. Then $t_n \in \mathscr{R}_{n,a}$ and

$$\begin{aligned} |t_n'(e^{i\theta_k})| &= |\mu a B_n'(e^{i\theta_k}) + (1-a)\, s_n'(e^{i\theta_k})| \\ &= a\,|B_n'(e^{i\theta_k})| + (1-a)\,|s_n'(e^{i\theta_k})| \\ &\qquad \text{(by choosing a suitable } \mu \in \mathbf{T}) \\ &\geqslant |B_n'(e^{i\theta_k})| - \frac{C}{m}(1-a), \end{aligned}$$

by (6.11). This completes our proof. ∎

Remark. We proved Theorem 3.3 by constructing a rational function which gave an upper bound of its derivative at specified points. Although we could not show that this rational function is extremal, we believe that it is pretty close to the extremal solution and the order of

$$\frac{(1-a)}{4\pi M_n}\left\{\frac{m_n}{M_n}(1-a) - \sin\frac{m_n}{M_n}(1-a)\right\}$$

in (3.2) is optimal as $n \to \infty$.

REFERENCES

[A] N. I. Achieser, "Theory of Approximation," Ungar, New York, 1956.

[BE] P. Borwein and T. Erdelyi, Sharp extensions of Bernstein inequality to rational spaces, *Mathematika* **43** (1996), 413–423.

[D] M. M. Djrbashian, A survey on the theory of orthogonal systems and some open problems, *in* "Orthogonal Polynomials: Theory and Practice" (P. Nevai, Ed.), pp. 135–146, Kluwer, Dordrecht, 1990.

[Du] P. Duren, "Theory of H_p Spaces," Academic Press, New York, 1969.

[FRS] C. Frappier, Q. I. Rahman, and St. Ruscheweyh, On polynomials with a prescribed zero, *Constr. Approx.* **2** (1986), 171–177.

[GR] A. Giroux and Q. I. Rahman, Inequalities for polynomials with a prescribed zero, *Trans. Amer. Math. Soc.* **193** (1974), 67–98.

[L] P. D. Lax, Proof of conjecture of P. Erdős on the derivative of a polynomial, *Bull. Amer. Math. Soc.* **50** (1944), 509–513.

[LMR] X. Li, R. N. Mohapatra, and R. S. Rodriguez, Bernstein-type inequalities for rational functions with prescribed poles, *J. London Math. Soc.* **51** (1995), 523–531.

[PP] P. P. Petrushev and V. A. Popov, "Rational Approximation of Real Functions," Encyclopedia of Mathematics and Its Applications, Vol. 28, Cambridge Univ. Press, Cambridge, UK, 1987.

[OW] P. E. Olivier and A. O. Watt, Polynomials with a prescribed zero and the Bernstein's inequality, *Canad. J. Math* **45** (1993), 627–637.

[RS] Q. I. Rahman and Stenger, An extremal problem for polynomials with a prescribed zero, *Proc. Amer. Math. Soc.* **43** (1974), 84–90.

[S] G. Szegö, "Orthogonal Polynomials," 4th ed., AMS Colloquium Publications, Vol. XXIII, American Mathematical Society, Providence, RI, 1975.

[T] P. Turán, "On a New Method of Analysis and Its Applications," Wiley, New York, 1984.

[W] J. L. Walsh, "Interpolation and Approximation by Rational Functions in the Complex Domain," AMS Colloquium Publications, Vol. XX, American Mathematical Society, Providence, RI, 1960.